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# Free Vibrations of Orthotropic Cylindrical Shell on Elastic Foundation

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## Nomenclature

$C_1, C_2, C_3$ = vibration amplitude constants in axial, tangential, and radial directions, respectively
$E_1, E_2$ = Young's moduli in $x$ and $\phi$ directions, respectively
$G$ = shear modulus
$G_{12}$ = modulus of rigidity associated with principal material directions
$\bar{G}$ = nondimensional shear modulus, $G(1-\nu_{12}\nu_{21})/E_1 h$
$h$ = shell thickness
$K$ = foundation modulus
$L$ = length of cylindrical shell
$m$ = number of axial half-waves
$n$ = number of circumferential waves
$R$ = radius of middle surface of the shell
$t$ = time coordinate
$U, V, W$ = natural vibrational modes in the longitudinal, circumferential, and radial directions, respectively
$u, v, w$ = displacement components of the middle surface in axial, circumferential, and radial directions, respectively ($w$-positive inward)
$\alpha^2$ = $E_2/E_1$
$\lambda$ = axial wave parameter, $\lambda^* R = m\pi R/L$
$\lambda^*$ = wave parameter, $m\pi/L$
$\bar{\mu}$ = nondimensional foundation parameter, $KR^2(1-\nu_{12}\nu_{21})/E_1 h$
$\nu_{12}, \nu_{21}$ = Poisson's ratios in $x$ and $\phi$ directions, respectively
$\rho$ = mass density of structural material
$\Omega$ = dimensionless frequency parameter
$\omega$ = angular frequency

## Introduction

CYLINDRICAL shells that have continuous and thorough contact with an elastic medium, solid or liquid, either on an outer or inner surface are considered as cylindrical shells on an elastic foundation. Such components and structures are often subjected to dynamic loads. Flow-induced vibrations in heat exchangers and pipelines, wave loading on submarines, vibrations of fuel-filled droptanks of fighter aircraft, underground and under-sea pipelines, and tunnels and semicircular roofs of underground aircraft hangers subject to seismic forces, nuclear explosions, and other blasts are some of the numerous examples.

Significant contributions have been made in the field of vibrations of cylindrical shells in general.[1–3] However, vibrations of isotropic shells on elastic foundations have been studied only recently.[4,5] Herein, the authors investigate the vibrations of an orthotropic cylindrical shell on an elastic foundation using membrane theory, inasmuch as the effect of the $h/R$ ratio on the eigenfrequencies of thin cylinders is insignificant. Response of elastic media is represented by Winkler and Pasternak foundation models (see Ref. 6). Winkler represented an elastic foundation by a set of closely spaced, independent linear springs. Pasternak interconnected these springs through a shear layer made of incompressible vertical elements.

## Analysis

Equations of motion for natural vibration of orthotropic circular cylindrical shell resting on a Pasternak foundation are as follows:

$$\frac{\partial^2 u}{\partial x^2} + \nu_{21}\frac{\partial}{\partial x}\left(\frac{\partial v}{R\partial\phi} - \frac{w}{R}\right) + \frac{G_{12}(1-\nu_{12}\nu_{21})}{E_1} \times \frac{\partial}{R\partial\phi}\left(\frac{\partial u}{R\partial\phi} + \frac{\partial v}{\partial x}\right) = \frac{\rho(1-\nu_{12}\nu_{21})}{E_1}\frac{\partial^2 u}{\partial t^2} \tag{1a}$$

$$\nu_{12}\alpha^2\frac{\partial^2 u}{R\partial\phi\partial x} + \alpha^2\frac{\partial}{R\partial\phi}\left(\frac{\partial v}{R\partial\phi} - \frac{w}{R}\right) + \frac{G_{12}(1-\nu_{12}\nu_{21})}{E_1} \times \frac{\partial}{\partial x}\left(\frac{\partial u}{R\partial\phi} + \frac{\partial v}{\partial x}\right) = \frac{\rho(1-\nu_{12}\nu_{21})}{E_1}\frac{\partial^2 v}{\partial t^2} \tag{1b}$$

$$\nu_{12}\alpha^2\frac{\partial u}{R\partial x} + \frac{\alpha^2}{R}\left(\frac{\partial v}{R\partial\phi} - \frac{w}{R}\right) = \frac{\rho(1-\nu_{12}\nu_{21})}{E_1}\frac{\partial^2 w}{\partial t^2} \times \frac{(1-\nu_{12}\nu_{21})}{E_1 h}(Kw - \bar{G}\nabla^2 w) \tag{1c}$$

Assuming that the cylindrical shell is simply supported at both ends, that is, $u(0)=0$, the general solution of Eq. (1) can be given by considering the boundary conditions in the form[7]

$$u = U(x)\cos n\phi\cos\omega t, \qquad v = V(x)\sin n\phi\cos\omega t$$
$$w = W(x)\cos n\phi\cos\omega t \tag{2}$$

where $U$, $V$, and $W$ can be expressed as

$$U = C_1\cos\lambda^* x, \qquad V = C_2\sin\lambda^* x, \qquad W = C_3\sin\lambda^* x \tag{3}$$

By the substitution of solutions (2) and (3) into partial differential equations (1), the three algebraic equations are obtained as

$$\left(\Omega^2 - H_1\right)C_1 + n\lambda[\nu_{21} + (G_{12}/E_1)(1-\nu_{12}\nu_{21})]C_2 - \nu_{21}\lambda C_3 = 0$$
$$n\lambda\left[(G_{12}/E_1)(1-\nu_{12}\nu_{21}) + \alpha^2\nu_{12}\right]C_1 + \left(\Omega^2 - H_2\right)C_2 + \alpha^2 n C_3 = 0$$
$$-\nu_{12}\lambda\alpha^2 C_1 + \alpha^2 n C_2 + \left(\Omega^2 - H_3\right)C_3 = 0 \tag{4}$$

For a nontrivial solution, the determinant of the coefficients of $C_1$, $C_2$, and $C_3$ must be zero. This yields the following bicubic equation for eigenfrequencies:

$$\bar{\Omega}^3 - \bar{\Omega}^2(H_1+H_2+H_3) + \bar{\Omega}\left[(H_1H_2 + H_2H_3 + H_3H_1) - \alpha^4 n^2 - \nu_{12}\nu_{21}\lambda^2\alpha^2 - n^2\lambda^2 P_1P_2\right] - \left[H_1H_2H_3 - n^2\alpha^4 H_1 - n^2\lambda^2 P_1P_2H_3 - \nu_{12}\nu_{21}\lambda^2\alpha^2 H_2 - \nu_{12}n^2\lambda^2\alpha^4 P_2 - \nu_{12}n^2\lambda^2\alpha^2 P_1\right] = 0 \tag{5}$$

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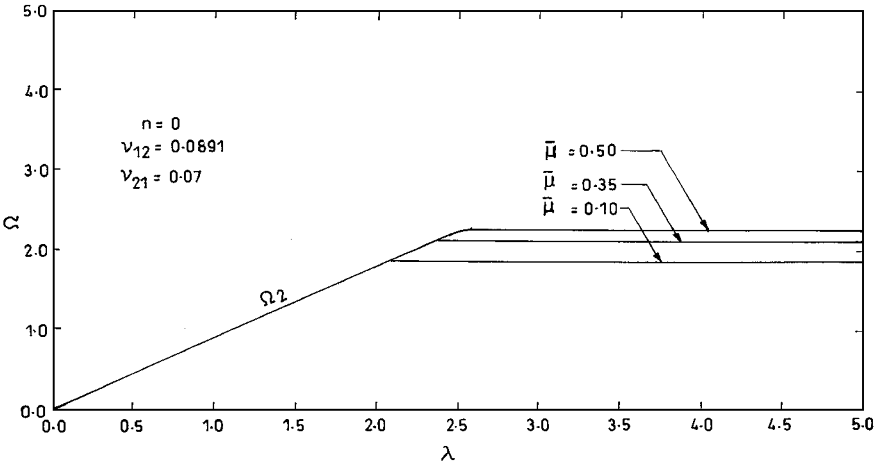


**Fig. 1 Variation of least eigenfrequency with respect to axial wave parameter $\lambda$ and nondimensional foundation parameter $\bar{\mu}$.**

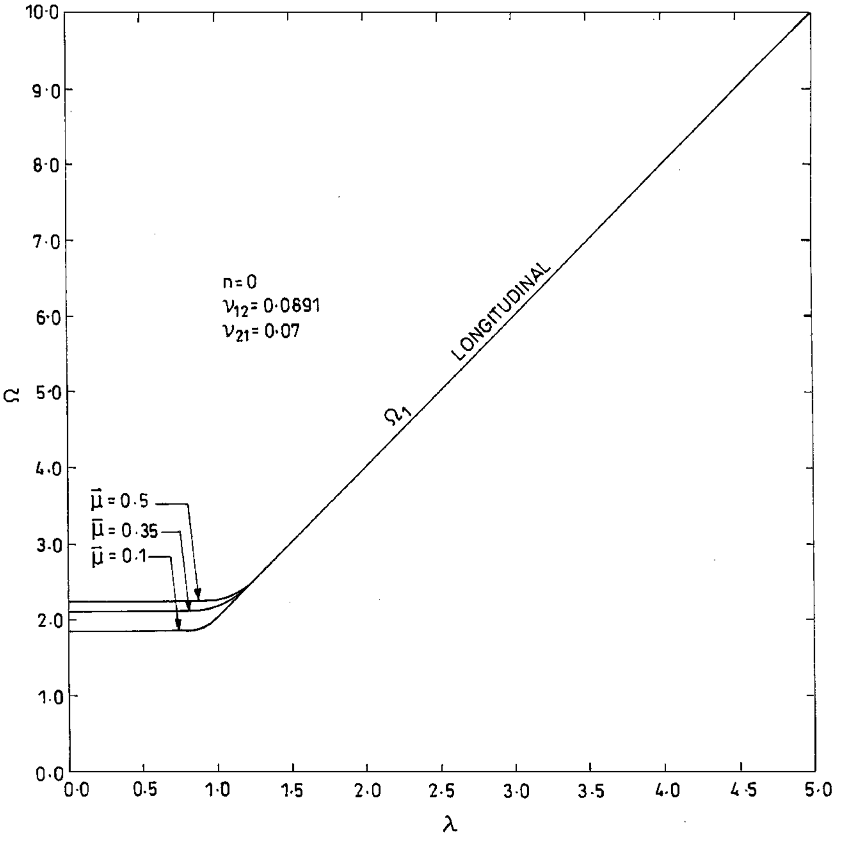


**Fig. 2 Variation of highest eigenfrequency with axial wave parameter $\lambda$ and nondimensional foundation parameter $\bar{\mu}$.**

where

$$P_1 = \frac{G_{12}}{E_1}(1 - \nu_{12}\nu_{21}) + \nu_{12}\alpha^2$$

$$P_2 = \frac{G_{12}}{E_1}(1 - \nu_{12}\nu_{21}) + \nu_{21}$$

$$H_1 = n^2\frac{G_{12}}{E_1}(1 - \nu_{12}\nu_{21}) + \lambda^2$$

$$H_2 = \lambda^2\frac{G_{12}}{E_1}(1 - \nu_{12}\nu_{21}) + \alpha^2 n^2$$

$$H_3 = \alpha^2 + \bar{\mu} + \bar{G}(\lambda^2 + n^2), \qquad \Omega^2 = \frac{\rho(1 - \nu_{12}\nu_{21})}{E_1} \cdot \omega^2 R^2$$

and $\bar{\Omega}$ stands for $\Omega^2$. From the bicubic eigenfrequency equation, it can be concluded that the eigenfrequencies of an orthotropic cylindrical shell depend on two mode numbers, that is, $n$, which controls the vibrational modes in circumferential direction of shell, and $\lambda$, which controls the vibrational modes in longitudinal direction. Equation (5) yields for each set of values of the axial and circumferential wave parameters, the nondimensional foundation modulus, and the nondimensional shear modulus, three frequencies that correspond to motion that is predominantly radial (lowest eigenfrequency), predominantly axial (largest eigenfrequency), and predominantly circumferential (middle eigenfrequency).

## Discussion

Figures 1–3, for $n = 0$, show that the nondimensional Winkler foundation modulus $\bar{\mu}$ affects the three eigenfrequencies ($\Omega_1$, $\Omega_2$, and $\Omega_3$) within that range of axial wave number $\lambda$, in which they represent the radial mode of vibration. The characteristics are very

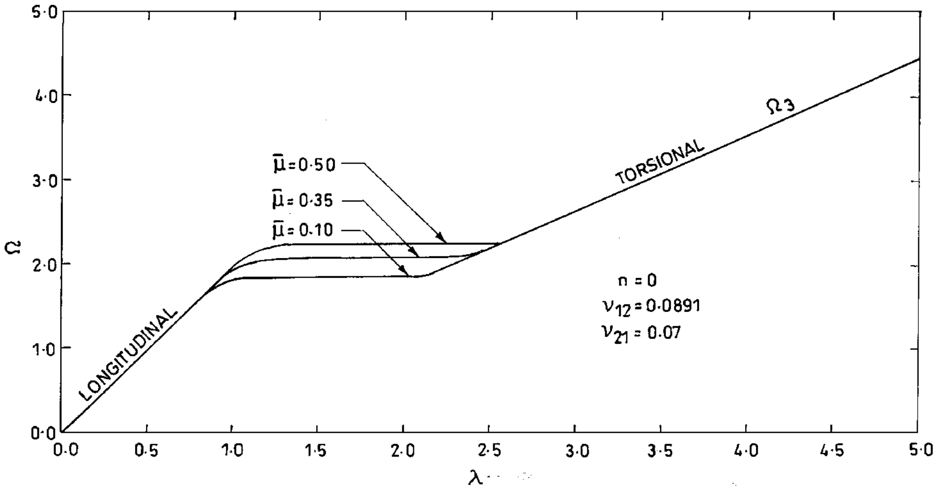


**Fig. 3 Variation of middle eigenfrequency with respect to axial wave parameter $\lambda$ and nondimensional foundation parameter $\bar{\mu}$.**

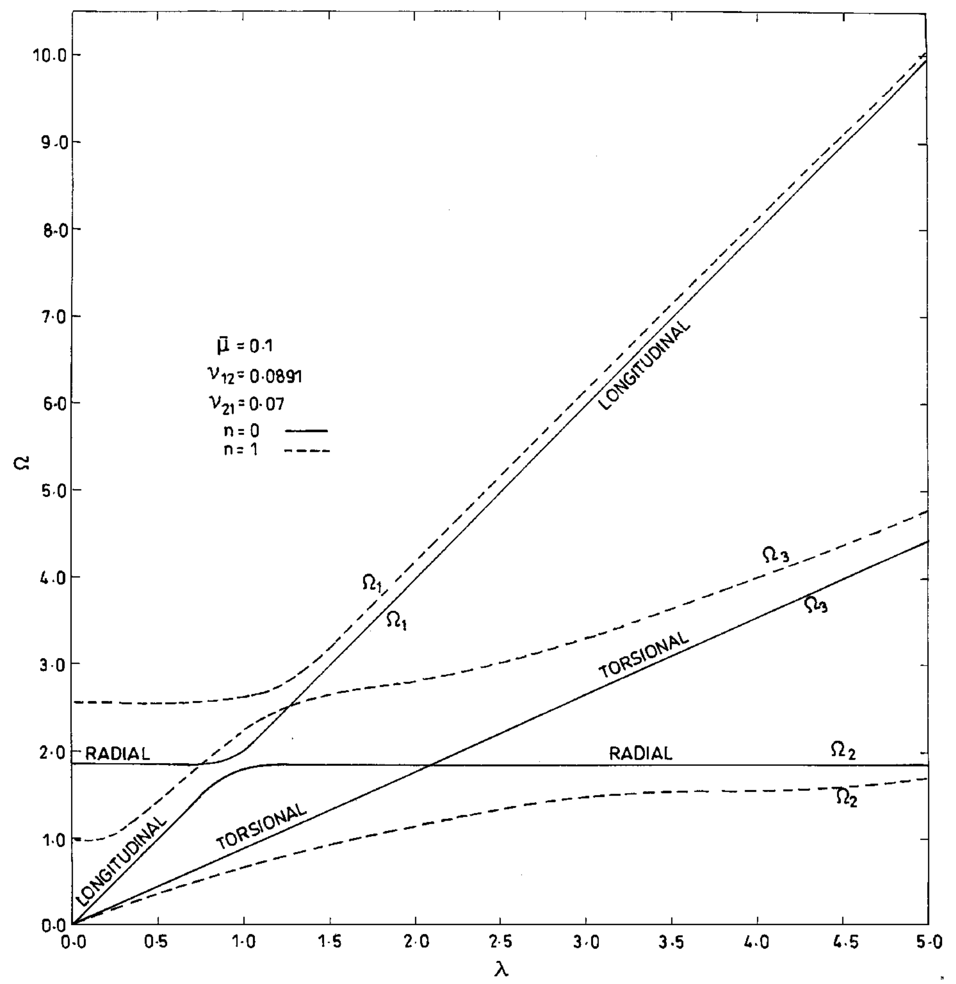


**Fig. 4 Variation of eigenfrequencies for $n = 0$ and 1.**

similar to those of an isotropic cylindrical shell on an elastic foundation.[4] Figure 4 shows $\Omega$ vs $\lambda$ curves for $n = 0$ and 1. For $n = 1$, the variation of the eigenfrequency $\Omega_1$ with $\lambda$ is the same for isotropic and orthotropic shells[4]; however, the variation of $\Omega_2$ and $\Omega_3$ with $\lambda$ do not match for isotropic[4] and orthotropic cylinders in the range $\lambda = 0$–1. The influence of the nondimensional shear modulus $\bar{G}$ of the Pasternak foundation on three eigenfrequencies is shown in Fig. 5. As soon as $\bar{G}$ is increased to 1, 2, or 3, $\Omega_2$ jumps to the value of $\Omega_3$ meant for the Winkler foundation ($\bar{G} = 0$) while $\Omega_3$ jumps to the value of $\Omega_1$ meant for the Winkler foundation. Frequency curves (for $n = 0$) for cylinders made of undirectional glass, undirectional carbon, Kevlar, and fabric, shown in Fig. 6, reveal that the largest eigenfrequency $\Omega_1$ remains unaffected by the elastic moduli ratio ($E_1/E_2$), whereas the middle frequency $\Omega_3$ and the lowest frequency $\Omega_2$ decrease with the increase in elastic moduli ratio.

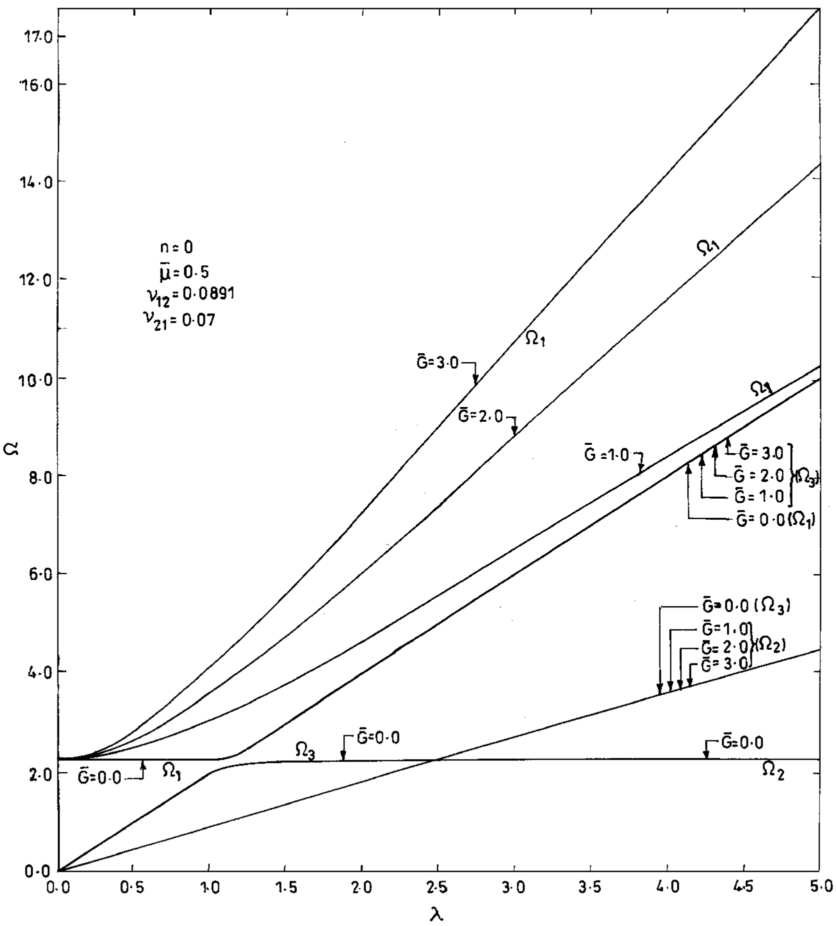


Fig. 5 Variation of eigenfrequencies with axial wave parameter $\lambda$ for different nondimensional foundation parameters $\bar{G}$.

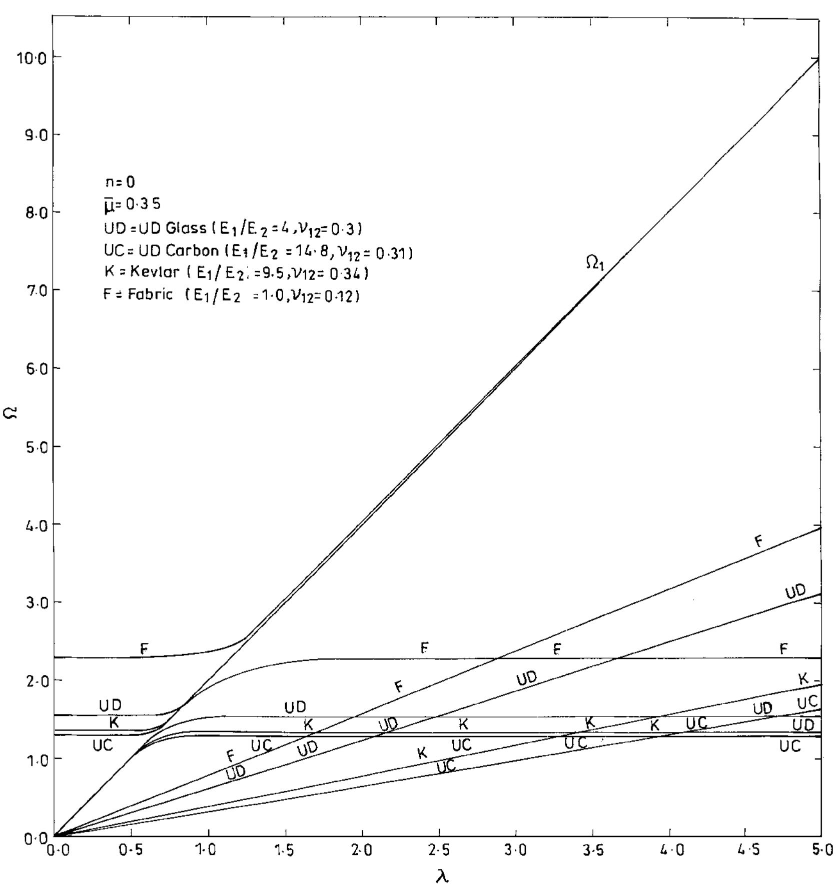


Fig. 6 Variation of eigenfrequencies with axial wave parameter $\lambda$ for various moduli ratio for $\bar{\mu} = 0.35$.

## Conclusion

1) The foundation parameters $\bar{\mu}$ and $\bar{G}$ play an important role in the dynamic characteristics of orthotropic circular cylindrical shells.

2) The foundation modulus $K$ considerably influenced the radial vibration frequency, whereas longitudinal and torsional vibration mode frequencies remained unaffected.

3) The foundation shear modulus $G$ affects all three vibration frequencies; however, its influence on the radial vibration mode is much more pronounced.

4) The change in elastic moduli ratio affects radial and torsional frequencies only; the longitudinal mode remains unaffected.

5) Vibrational characteristics are greatly influenced by the shell geometry.

6) An interesting feature of the axisymmetric case is that the torsional frequency is always less than the axial vibration frequency.

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